

Walking into an Absolute Sum

Hans J. H. Tuenter

Schulich School of Business, York University,
Toronto, Ontario, Canada M3J 1P3
email: htuent@schulich.yorku.ca

November 23, 1999

1 Introduction

Recently, it was asked by Paul Bruckman [1] to show that the sum

$$S_r(n) = \sum_{k=0}^{2n} \binom{2n}{k} |n - k|^r \quad (1)$$

evaluates to $n^2 \binom{2n}{n}$ for $r = 3$. In the published solution [16], it was also noted that $S_1(n) = n \binom{2n}{n}$, and, as a consequence, it was conjectured that $S_{2r+1}(n)$ equals the product of $\binom{2n}{n}$ and a monic polynomial of degree $r + 1$.

We show this conjecture to be true, albeit with the modification of discarding the adjectival modifier “monic.” In fact, we show that

$$S_{2r+1}(n) = P_r(n) n \binom{2n}{n} \quad \text{and} \quad S_{2r}(n) = Q_r(n) 2^{2n-r},$$

where $P_r(n)$ and $Q_r(n)$ are both polynomials of degree r with integer coefficients. We then investigate the relationship of these polynomials to the Dumont-Foata polynomials [6]. These are generalizations of the Gandhi polynomials, which find their origin in a representation of the Genocchi numbers, first conjectured by Gandhi [9]. Finally, we show that the sums $S_r(n)$ are essentially the moments of a random variate, measuring the absolute distance to the origin in a symmetric Bernoulli random walk, after $2n$ time steps.

2 Derivation

We note that the sum can be rewritten as

$$S_r(n) = 2 \sum_{k=0}^n \binom{2n}{n-k} k^r - \binom{2n}{n} \delta_{r0},$$

with δ_{r0} the Kronecker delta. Now consider, for $r \geq 1$,

$$n^2 S_r(n) - S_{r+2}(n) = 2 \sum_{k=0}^{n-1} \binom{2n}{n-k} k^r (n^2 - k^2) = 4n(2n-1) \sum_{k=0}^{n-1} \binom{2n-2}{n-1-k} k^r,$$

leading directly to the recursion

$$S_{r+2}(n) = n^2 S_r(n) - 2n(2n-1) S_r(n-1). \quad (2)$$

For $r = 0$ the derivation is slightly more elaborate, because we need to keep track of the additional term, but leads to the same recursion, so that (2) is valid for all nonnegative integers r . To start the recursion, we find the value $S_0(n) = 2^{2n}$ by an application of the binomial theorem to (1). The value of $S_1(n)$ is easily obtained by breaking up the summand k to create two sums:

$$S_1(n) = \sum_{k=0}^n \binom{2n}{n-k} [(n+k) - (n-k)] = 2n \sum_{k=0}^n \binom{2n-1}{n-k} - 2n \sum_{k=0}^{n-1} \binom{2n-1}{n-k-1},$$

and one sees that, after changing the range of summation of the second sum to start at $k = 1$, all terms cancel out, with the exception of the summand $2n \binom{2n-1}{n}$. Rearranging terms gives the desired $S_1(n) = n \binom{2n}{n}$.

It is now clear that the structure of the sum depends upon the parity of r . Starting with the odd values, we simplify the recursion (2) by the substitution $S_{2r+1}(n) = P_r(n) n \binom{2n}{n}$ to give

$$P_{r+1}(n) = n^2 [P_r(n) - P_r(n-1)] + nP_r(n-1), \quad (3)$$

with initial condition $P_0(n) = 1$. An inductive argument now shows that $P_r(n)$ is a polynomial of degree r with integer coefficients, and proves the modified conjecture. It is not difficult to show that $r!$ is the leading coefficient of $P_r(n)$, and, hence, that these polynomials are not monic. In fact, the only cases for which the leading coefficient is 1 are $r = 0$ and $r = 1$. The first few polynomials are now easily determined as:

$$\begin{aligned} P_0(n) &= 1, \\ P_1(n) &= n, \\ P_2(n) &= (2n-1)n, \\ P_3(n) &= (6n^2 - 8n + 3)n, \\ P_4(n) &= (24n^3 - 60n^2 + 54n - 17)n, \\ P_5(n) &= (120n^4 - 480n^3 + 762n^2 - 556n + 155)n. \end{aligned}$$

For the even sums we substitute $S_{2r}(n) = Q_r(n) 2^{2n-r}$ to give the recursion

$$Q_{r+1}(n) = 2n^2 [Q_r(n) - Q_r(n-1)] + nQ_r(n-1), \quad (4)$$

with initial condition $Q_0(n) = 1$. This shows that $Q_r(n)$ is a polynomial of degree r with integer coefficients. It is not difficult to establish that the leading coefficient is given by $(2r-1) \cdot (2r-3) \cdots 3 \cdot 1 = (2r)!/(2^r r!)$, and, hence, that these polynomials are also not monic. Applying the recursion gives the first few polynomials as:

$$\begin{aligned} Q_0(n) &= 1, \\ Q_1(n) &= n, \\ Q_2(n) &= (3n-1)n, \\ Q_3(n) &= (15n^2 - 15n + 4)n, \\ Q_4(n) &= (105n^3 - 210n^2 + 147n - 34)n, \\ Q_5(n) &= (945n^4 - 3150n^3 + 4095n^2 - 2370n + 496)n. \end{aligned}$$

It is worth noting that, by evaluating $S_r(n)$ for particular values of n , one can derive various properties of [the coefficients of] the polynomials $P_r(n)$ and $Q_r(n)$. For instance, it is not difficult to show that the coefficients of $P_r(n)$ sum to unity, and those of $Q_r(n)$ to 2^{r-1} (for $r \geq 1$), by evaluating the sums for $n = 1$. Indeed, one can derive the closed-form solutions for $S_{2r}(n)$ and

$S_{2r+1}(n)$, by solving a system of linear equations in r unknowns, representing the coefficients of the corresponding polynomial.

In the constant of the polynomials $P_r(n)/n$ one recognizes the Genocchi numbers [4, 10], named after the Italian mathematician Angelo Genocchi (1817–1889):

$$G_2 = -1, \quad G_4 = 1, \quad G_6 = -3, \quad G_8 = 17, \quad G_{10} = -155, \quad G_{12} = 2073, \quad \dots$$

These integers are defined through the exponential generating function

$$\frac{2t}{e^t + 1} = t + \sum_{r \geq 1} G_{2r} \frac{t^{2r}}{(2r)!},$$

and are related to the Bernoulli numbers by $G_{2r} = 2(1 - 2^{2r})B_{2r}$. The Genocchi numbers are listed as sequence A001469 in the on-line version of the encyclopedia of integer sequences [15], where additional references may be found. The constant of the polynomials $Q_r(n)/n$ matches the first terms of the sequence A002105 in [15], which is generated by $2^{r-1}G_{2r}/r$, and related to the tangent numbers. The connection to the Genocchi numbers will be further explored in the next section, where the polynomials $P_r(n)$ and $Q_r(n)$ are found to be related to special cases of the Dumont-Foata polynomials.

Another matter of interest is the leading coefficient of the polynomials, characterizing the behavior of the sums $S_r(n)$ for large values of n . For the even indexed sums, this is easily established as

$$S_{2r}(n) \sim \frac{(2r)!}{2^{2r} r!} 2^{2n} n^r, \quad (5)$$

and for the odd indexed sums we can use Stirling's formula to give $\binom{2n}{n} \sim 2^{2n}/\sqrt{\pi n}$, so that

$$S_{2r+1}(n) \sim \frac{r!}{\sqrt{\pi}} 2^{2n} n^{r+\frac{1}{2}}. \quad (6)$$

In these expressions, one can recognize the moments of a central chi-distribution, see for instance [12, pp. 420–421]. That this is no coincidence will be shown in section 4 where we establish the connection between the sums $S_r(n)$ and the distance to the origin in a symmetric Bernoulli random walk.

3 Dumont-Foata polynomials

In this section we show that the polynomials $P_r(n)$ and $Q_r(n)$ are related to special cases of the Dumont-Foata polynomials [6]. These are defined recursively by means of

$$F_{r+1}(x, y, z) = (x + z)(y + z)F_r(x, y, z + 1) - z^2 F_r(x, y, z),$$

with initial condition $F_1(x, y, z) = 1$. Explicit expressions for these polynomials and their generating functions have been derived by Carlitz [3], but are too lengthy to display here.

The Dumont-Foata polynomials can be regarded as generalizations of the Gandhi polynomials, see for instance [5, 17], which are defined by the recursion

$$\tilde{P}_{r+1}(z) = (z + 1)^2 \tilde{P}_r(z + 1) - z^2 \tilde{P}_r(z),$$

with initial condition $\tilde{P}_1(z) = 1$. The coefficients of the first few of these polynomials are displayed in Table 1, and can also be found in [15, Seq. A036970]. The Gandhi polynomials arose from a conjecture made by Gandhi [9], concerning a representation of the Genocchi numbers. Gandhi's conjecture that $\tilde{P}_r(0) = (-1)^r G_{2r}$ was proved by Carlitz [2], and also by Riordan and Stein [14].

				1					
				2		1			
		6		8		3			
	24		60		54		17		
120		480		762		556		155	
720		4200		10248		12840		8146	2073

Table 1: Coefficients of the Gandhi polynomials, arranged in triangular form.

					1				
				3		1			
		15		15		4			
	105		210		147		34		
945		3150		4095		2370		496	
10395		51975		107415		111705		56958	11056

Table 2: Coefficients of the polynomials $\tilde{Q}_r(z)$, arranged in triangular form.

Another polynomial that can be derived as a special case of the Dumont-Foata polynomials is given by $\tilde{Q}_r(z) = 2^{r-1} F_r(\frac{1}{2}, 1, z)$, and is generated by the recursion

$$\tilde{Q}_{r+1}(z) = (2z + 1)(z + 1)\tilde{Q}_r(z + 1) - 2z^2\tilde{Q}_r(z),$$

with initial condition $\tilde{Q}_1(z) = 1$. The coefficients of the first few of these polynomials are displayed in Table 2. For $r \geq 1$, one can easily verify by substitution in (3) and (4) that $P_r(n) = (-1)^{r-1} n \tilde{P}_r(-n)$ and $Q_r(n) = (-1)^{r-1} n \tilde{Q}_r(-n)$. This gives the connection to the Dumont-Foata polynomials (for positive r) as:

$$P_r(n) = (-1)^{r-1} n F_r(1, 1, -n)$$

and

$$Q_r(n) = (-2)^{r-1} n F_r(\frac{1}{2}, 1, -n).$$

The occurrence of the Genocchi numbers in the constant of the polynomials $P_r(n)/n$ is now seen to be a direct consequence of Gandhi's conjecture that $F_r(1, 1, 0) = (-1)^r G_{2r}$. The occurrence of the Genocchi numbers in the constant of the polynomials $Q_r(n)/n$ is conjectured by the present author in the form $F_r(\frac{1}{2}, 1, 0) = (-1)^r G_{2r}/r$.

4 Symmetric Bernoulli random walks

In a symmetric Bernoulli random walk, one considers the movements of a particle starting at time $t = 0$ at the origin. Its movements are determined by a chance mechanism, where a fair coin is flipped and the particle is moved one unit to the right if it is heads up, and one unit to the left if it is tails up. A more exhaustive description and in-depth study of random walks can be found in Feller [8] or Révész [13]. A more playful introduction to the topic is given in the monograph by Dynkin and Uspenskii [7]. A topic of interest is the position of the particle after $2n$ coin tosses: $Y_{2n} = X_1 + X_2 + \cdots + X_{2n}$, where X_i is $+1$ or -1 depending upon whether or not the coin showed heads in the i th coin toss. Note that the X_i are independent and identically distributed variates with mean 0 and variance 1. The probability distribution of the position of the particle after $2n$ moves can be derived from a simple combinatorial argument, see for instance [8, p. 75] or [13, p. 13], and is given by

$$\text{Prob}(Y_{2n} = 2k) = \binom{2n}{n-k} 2^{-2n},$$

where $k = -n, -n+1, \dots, n$, and n a positive integer. The matter of interest in the context of this note is the distance to the origin, $|Y_{2n}|$ at time $t = 2n$. Its moments are given by

$$\mathbb{E} |Y_{2n}|^r = \sum_{k=-n}^n \binom{2n}{n-k} 2^{-2n} |2k|^r,$$

and one sees that $\mathbb{E} |Y_{2n}|^r = 2^{r-2n} S_r(n)$, thus establishing the connection to the absolute sums from the introduction. The limit behavior of these sums now becomes clear. By the central limit theorem, see for instance [11, p. 18], one has that Y_{2n} , for sufficiently large n , follows a normal distribution with mean 0 and variance $2n$. This implies that asymptotically, $|Y_{2n}|$ has a half-normal or central chi-distribution, so that

$$\mathbb{E} |Y_{2n}|^r \sim \frac{\Gamma[(r+1)/2]}{\Gamma(1/2)} 2^r n^{r/2},$$

see for instance [12, pp. 420–421]. This gives the asymptotic behavior of the sums as

$$S_r(n) = 2^{2n-r} \mathbb{E} |Y_{2n}|^r \sim \frac{\Gamma[(r+1)/2]}{\Gamma(1/2)} 2^{2n} n^{r/2},$$

and upon expanding the gamma functions one recovers the limit results (5) and (6).

5 Discussion

One could possibly use the relation of the Gandhi polynomials to the sums $S_{2r+1}(n)$ to gain new insight into the former. In particular, one now has an expression to derive the function values of the Gandhi polynomials for negative, integral arguments:

$$\tilde{P}_r(-n) = (-1)^{r-1} \frac{2}{n^2} \binom{2n}{n}^{-1} \sum_{k=1}^n \binom{2n}{n-k} k^{2r+1}.$$

For example, one easily obtains $\tilde{P}_r(-1) = (-1)^{r-1}$ and $\tilde{P}_r(-2) = (-1)^{r-1} (2^{2r-1} + 1)/3$.

Likewise, one can use the relation of the moments of the absolute distance to the origin in a symmetric Bernoulli random walk and the sums $S_r(n)$ to express these moments in terms of the polynomials $P_r(n)$ and $Q_r(n)$:

$$\mathbb{E} |Y_{2n}|^{2r} = 2^r Q_r(n) \quad \text{and} \quad \mathbb{E} |Y_{2n}|^{2r+1} = \binom{2n}{n} 2^{2(r-n)+1} n P_r(n).$$

This equivalence can be used to establish the rate of convergence to the moments of the half-normal distribution.

Finally, it should be noted that one can also determine expressions for $S_{2r}(n)$ by means of the generating function

$$f_n(\varphi) = \sum_{k=0}^{2n} \binom{2n}{k} e^{(n-k)\varphi} = e^{n\varphi} [1 + e^{-\varphi}]^{2n} = 2^n [1 + \cosh \varphi]^n,$$

so that $S_{2r}(n) = f_n^{(2r)}(0)$. However, this approach covers only the even indexed case, and does not give the same insight into the problem as the one that we have followed here.

6 Acknowledgment

I would like to thank the anonymous referee for drawing attention to the occurrence of the Genocchi numbers in the polynomials $P_r(n)$. This led to a further investigation and the characterization in terms of the Gandhi and Dumont-Foata polynomials.

References

- [1] Paul S. Bruckman. Problem B-871. *The Fibonacci Quarterly*, 37(1):85, February 1999.
- [2] L. Carlitz. A conjecture concerning Genocchi numbers. *Det Kongelige Norske Videnskabers Selskabs Skrifter*, 9:1–4, 1972.
- [3] L. Carlitz. Explicit formulas for the Dumont-Foata polynomial. *Discrete Mathematics*, 30(3):211–225, 1980.
- [4] Louis Comtet. *Advanced combinatorics; the art of finite and infinite expansions*. D. Reidel, Boston, 1974.
- [5] Dominique Dumont. Interpretations combinatoires des nombres de Genocchi. *Duke Mathematical Journal*, 41:305–318, 1974.
- [6] Dominique Dumont and Dominique Foata. Une propriété de symétrie des nombres de Genocchi. *Bulletin de la Société Mathématique de France*, 104(4):433–451, 1976.
- [7] E. B. Dynkin and V. A. Uspenskii. *Random Walks: Part Three of Mathematical Conversations*. D. C. Heath and Company, Boston, 1963.
- [8] William Feller. *An Introduction to Probability Theory and Its Applications*. John Wiley & Sons, Inc., New York, 3rd edition, 1950.
- [9] J. M. Gandhi. A conjectured representation of Genocchi numbers. *The American Mathematical Monthly*, 77(5):505–506, May 1970.
- [10] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete Mathematics*. Addison-Wesley, Reading, Massachusetts, 1989.
- [11] Allan Gut. *Stopped Random Walks*. Springer-Verlag, New York, 1988.
- [12] Norman L. Johnson, Samuel Kotz, and N. Balakrishnan. *Continuous Univariate Distributions*, volume I. John Wiley & Sons, Inc., New York, 1994.
- [13] Pál Révész. *Random Walk in Random and Non-Random Environments*. World Scientific Publishing Co., Singapore, 1990.
- [14] John Riordan and Paul R. Stein. Proof of a conjecture on Genocchi numbers. *Discrete mathematics*, 5(4):381–388, 1973.
- [15] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences*, 2000. Published electronically at <http://www.research.att.com/~njas/sequences/>.
- [16] Indulis Strazdins. Solution to problem B-871. *The Fibonacci Quarterly*, 38(1):86–87, February 2000.
- [17] Volker Strehl. Alternating permutations and modified Ghandi-polynomials. *Discrete Mathematics*, 28:89–100, 1979.

AMS Classification Numbers: 11B65, 60G50, 44A60